

FORMAL SOLUTIONS FOR THE POLARIZED RADIATIVE TRANSFER EQUATION

Gioele Janett, Oskar Steiner, and Luca Belluzzi

Istituto Ricerche Solari Locarno

The task to numerically compute reliable and accurate emergent Stokes profiles is of great relevance in solar physics. Aiming at facilitating the comprehension of the advantages and drawbacks of the different formal solvers, this work presents a reference paradigm for their characterization based on the concepts of order of accuracy, stability, and computational cost.

Radiative Transfer Equation

The radiative transfer equation for polarized light is usually written under the form

$$\frac{d}{ds} \mathbf{I}(s) = -\mathbf{K}(s)\mathbf{I}(s) + \boldsymbol{\epsilon}(s),$$

where $\mathbf{I} = (I, Q, U, V)$ is the Stokes vector. The 4×4 absorption matrix \mathbf{K} and the emission vector $\boldsymbol{\epsilon} = (\epsilon_I, \epsilon_Q, \epsilon_U, \epsilon_V)$ are known at a discrete set of spatial points only.

Numerical schemes

Year	Method	Proposed by
1974	Runge-Kutta-Merson	[Wittmann, 1974]
1976	Runge-Kutta 4	[Landi Degl'Innocenti, 1976]
1985	Evolution Operator	[Landi Degl'Innocenti and Landi Degl'Innocenti, 1985]
1989	Zeeman Feautrier	[Rees et al., 1989]
1989	DELO-linear	[Rees et al., 1989]
1998	(cubic) Hermitian	[Bellot Rubio et al., 1998]
1999	DIAGONAL	[López Ariste and Semel, 1999]
2003	DELOPAR	[Trujillo Bueno, 2003]
2013	DELO-Bézier	[De la Cruz Rodríguez and Piskunov, 2013]
2013	BESSER	[Štěpán and Trujillo Bueno, 2013]
2016	Piecewise Continuous	[Steiner et al., 2016]

Order of accuracy

Consider the numerical approximation \mathbf{I}_k of the exact value $\mathbf{I}(s_k)$ for $k = 0, \dots, N$.

The global error is defined as

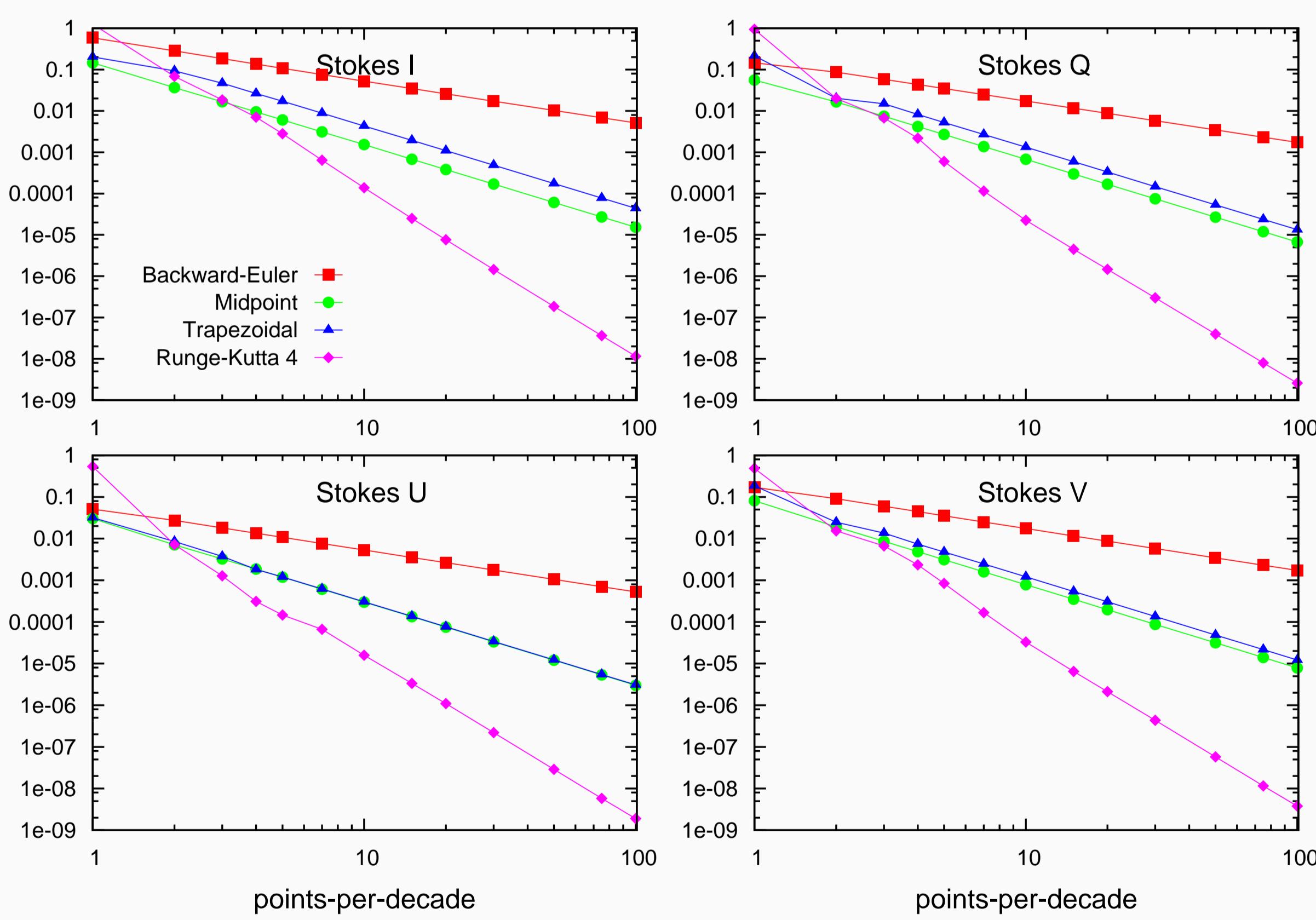
$$E_N = \|\mathbf{I}_N - \mathbf{I}(s_N)\|.$$

A numerical method for an ODE has order of accuracy p if

$$E_N = C \cdot \Delta s^p + O(\Delta s^{p+1}), \text{ with } p > 0,$$

where C is a constant. The log – log plot shows slope p

$$\log(E_N) \approx p \log(\Delta s) + \tilde{C}.$$



Conclusions

- Clear characterization of existing formal solvers.
- Identification of practical instability issues.
- Definition of optimal formal solvers.

Stability

Stability guarantees that any perturbation is not magnified by the numerical method.

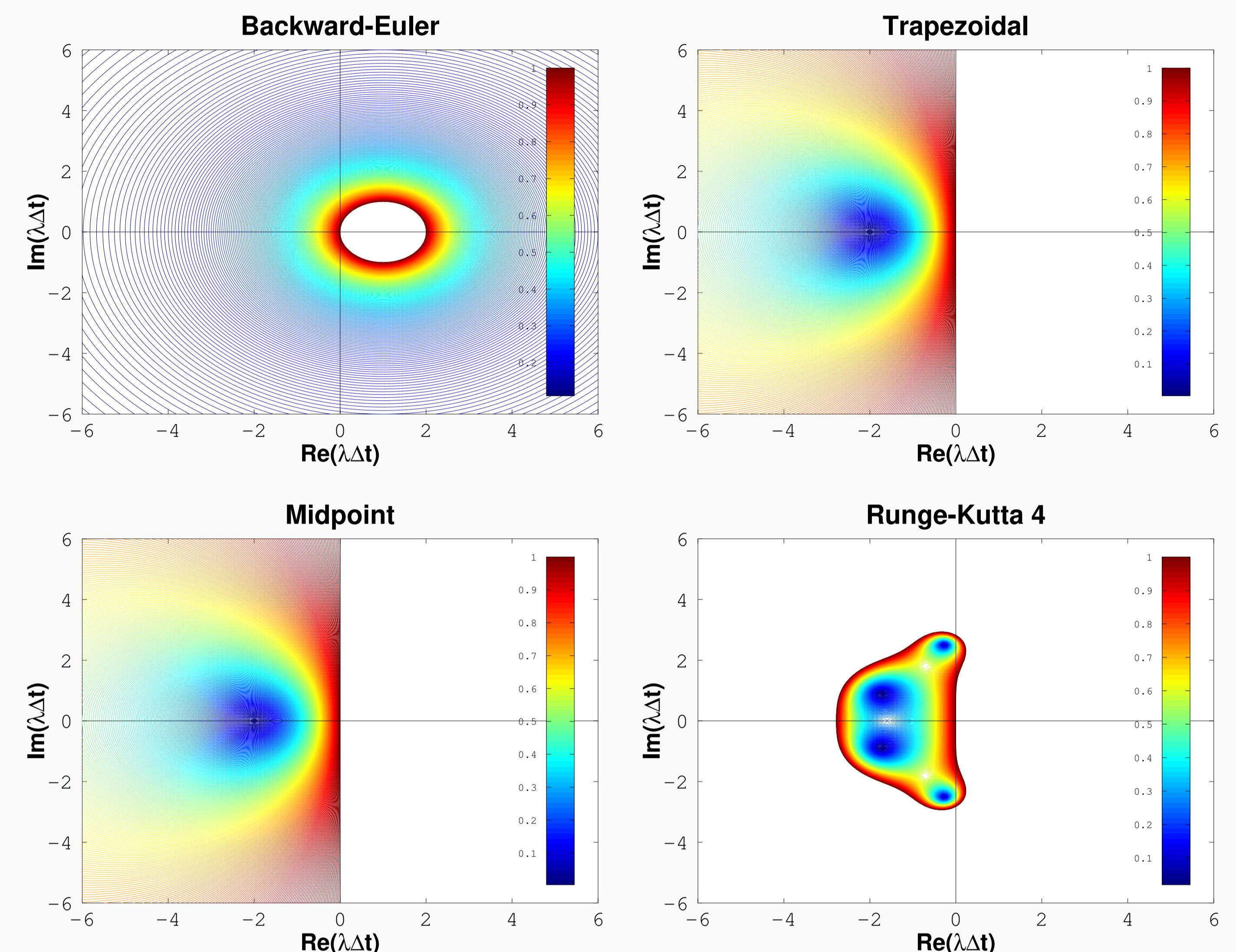
Consider the model ODE:

$$y'(t) = \lambda y(t) \Rightarrow y(t) = y_0 e^{\lambda t} \Rightarrow \lim_{t \rightarrow \infty} y(t) = 0 \text{ for } \operatorname{Re}(\lambda) < 0.$$

The numerical method $y_{k+1} = \phi(\lambda \Delta t) y_k$ is said to be stable if

$$\|\phi(\lambda \Delta t)\| < 1.$$

The stability region is the set of complex values of $\lambda \Delta t$ for which a numerical method is stable.



- A-stability: the stability region contains the whole left-hand side of the complex plane.
- L-stability: A-stability with the further condition

$$\lim_{\operatorname{Re}(z) \rightarrow -\infty} \phi(z) = 0.$$

- Variations in λ (i.e., $\lambda = \lambda(t)$) modify the stability conditions.

Computational cost

Computational cost depends on the complexity of the algorithm.

Explicit methods

$$y_{k+1} = f(y_k)$$

Implicit methods

$$g(y_{k+1}, y_k) = 0$$

- Explicit methods are more efficient (factor 2-5).
- Explicit methods often show instability issues.
- High-order methods are more costly per step.

Future topics

- Behaviour of numerical methods with highly intermittent atmosphere models.
- Behaviour of numerical methods with discontinuous atmosphere models.
- 2D and 3D radiative transfer problems.

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